# The Large-Time Solution of Burgers' Equation with Time- Dependent Coefficients. II. Algebraic Coefficients

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# The large-time solution of Burgers' equation with time dependent coefficients. II. Algebraic coefficients.

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**Abstract**

In this paper, we consider an initial-value problem for Burgers' equation with variable coefficients

$$u_t + \Phi(t)\, uu_x = \Psi(t)\, u_{xx}, \quad -\infty < x < \infty, \quad t > 0,$$

where $x$ and $t$ represent dimensionless distance and time respectively whilst $\Psi(t)$, $\Phi(t)$ are given continuous functions of $t$ $(> 0)$. In particular, we consider the case when the initial data has algebraic decay as $|x| \to \infty$, with $u(x,t) \to u_+$ as $x \to \infty$ and $u(x,t) \to u_-$ as $x \to -\infty$. The constant states $u_+$ and $u_-$ $(\neq u_+)$ are problem parameters. We focus attention on the case when $\Phi(t) = t^\delta$ (with $\delta > -1$) and $\Psi(t) = 1$. The method of matched asymptotic coordinate expansions is used to obtain the large-$t$ asymptotic structure of the solution to the initial-value problem over all parameter values.

## 1 Introduction

In this paper we consider the following initial-value problem for Burgers' equation with variable coefficients, namely,

$$u_t + \Phi(t)\, uu_x = \Psi(t)\, u_{xx}, \quad -\infty < x < \infty, \quad t > 0, \tag{1.1}$$

$$u(x,0) = u_0(x), \quad -\infty < x < \infty, \tag{1.2}$$

$$u(x,t) \to \begin{cases} u_-, & x \to -\infty, \\ u_+, & x \to \infty, \end{cases} \quad t \geq 0, \tag{1.3}$$

where $u_-$ and $u_+ (\neq u_-)$ are parameters and the functions $\Phi(t)$ and $\Psi(t)$ are algebraic functions of $t$. Further, we assume that the initial data $u_0(x)$ is continuously differentiable and has algebraic decay as $|x| \to \infty$. Specifically,

$$u_0(x) = \begin{cases} u_- + \dfrac{A_L}{(-x)^\gamma} + O\Big(E(|x|)\Big) & \text{as} \quad x \to -\infty, \\ u_+ + \dfrac{A_R}{x^\gamma} + O\Big(E(|x|)\Big) & \text{as} \quad x \to \infty, \end{cases} \tag{1.4}$$

where $A_L\,(\neq 0), A_R\,(\neq 0)$ and $\gamma\,(> 0)$ are parameters and $E(|x|)$ is linearly exponentially small in $x$ as $|x| \to \infty$. In particular when $u_+ > u_-$ then we take $A_L > 0$ and $A_R < 0$, whereas when $u_+ < u_-$ then we take $A_L < 0$ and $A_R > 0$. In what follows we refer to initial-value problem (1.1)-(1.4) as **IVP**.

In part one of this series of papers, [3] (hereafter referred to as (I)), a general introduction to **IVP** is given which reviews the literature relating to equation (1.1) as well as examining the relationship between equation (1.1) and the generalized Burger's equation

$$u_t + uu_x + f(t)u = u_{xx}, \tag{1.5}$$

for various functions, $f(t)$, and this is not repeated here. However, we recall that equations of the form (1.5) describe the propagation of waves in a gas or liquid. These waves are not only subject to diffusive and convective effects, but also geometric effects. For example, when $f(t) = \frac{\lambda}{t}$ where $\lambda$ ($\in (-1, \infty)$) is a constant, equation (1.5) models the propagation of finite-amplitude sound waves in variable ducts (see [7] where here $x$ represents time whilst $t$ represents the distance along the duct). On transforming equation (1.5), in this case to the form of equation (1.1), we find that $\Psi(t) = 1$ and $\Phi(t) = t^\delta$ where $\delta = -\frac{\lambda}{(\lambda+1)}$ ($\in (-1, \infty)$) is related to the duct cross-sectional area.

In (I) the method of matched asymptotic coordinate expansions (see for example [4], [5] and [9]) is used to obtain the complete large-time solution of **IVP** when:

(i) $\Phi(t) = e^t, \quad \Psi(t) = 1.$

(ii) $\Phi(t) = 1, \quad \Psi(t) = e^t.$

The form of the large-time solution of **IVP** in case (i) depends on the problem parameters $u_+$ and $u_-$. Specifically, when $u_+ > u_-$ the solution of **IVP** exhibits the formation of an expansion wave, whilst when $u_+ < u_-$ the solution of **IVP** exhibits the formation of a localized Taylor shock profile. In both cases the large-time attractor, the expansion wave or localized Taylor shock profile, connects $u = u_+$ to $u = u_-$. The form of the large-time solution of **IVP** in case (ii) is independent of the problem parameters and exhibits the formation of an error function profile connecting $u = u_+$ to $u = u_-$. In the current paper we will extend (I) by considering, without loss of generality, the case when

(iii) $\Phi(t) = t^\delta \ \big(\delta > -1\big), \quad \Psi(t) = 1.$

The more general situation of equation (1.1) with $\Phi(t) = t^\alpha$ ($\alpha > -1$) and $\Psi(t) = t^\beta$ ($\beta > -1$) where $\alpha \neq \beta$ can be transformed to case (iii) by the change of variables

$$u = (\beta + 1)^{-\delta}\bar{u}, \quad \tau = \int_0^t s^\beta \, \mathrm{d}s,$$

where $\delta = \frac{\alpha-\beta}{\beta+1}$ ($\in (-1, \infty)$). When $\alpha = \beta$ the change of variable $\tau = \int_0^t s^\alpha \, \mathrm{d}s$ transforms (1.1) to the classical Burgers' equation. It is instructive to note that equation (1.1) with $\Phi(t) = t^{-1}$ and $\Psi(t) = 1$ ($\Phi(t) = 1$ and $\Psi(t) = t^{-1}$) can be transformed to case (ii) ((i)) of (I) by the change of variable $\tau = \ln t$, respectively. Specifically, the method of matched asymptotic coordinate expansions is used to obtain the complete large-time solution to **IVP**. We have shown in Section 2 that, depending on the problem parameters $\delta$, $\gamma$, $u_+$ and $u_-$, a range of large-time attractors can arise in the solution to **IVP**. This range of large-time attractors includes: expansion wave, Taylor shock (hyperbolic tangent), Rudenko Soluyan similarity solution and error function profile. Table 1 provides an overview of the results contained within this paper, presenting which large-time attractor arises in the solution to **IVP** for the given problem parameters. The full details of these results are given in Section 3. This paper illustrates for the first time the connection between the initial data, problem parameters and the resulting large-time attractor for **IVP**.

Finally, we conclude by noting that the specific case when $\delta = 0$ is not considered in this paper. However, the interested reader is referred to [2] where the large-time solution of initial and initial-boundary value problems for the constant coefficient Burgers' equation is considered in detail.

| | $u_+ < u_-$ | $u_+ > u_-$ |
|---|---|---|
| $\delta > -\frac{1}{2}$ | Taylor Shock | Expansion Wave |
| $\delta = -\frac{1}{2}$ | Rudenko-Soluyan Similarity Solution | |
| $-1 < \delta < -\frac{1}{2}$ | Error Function | |

Table 1: The type of large-time attractor connecting $u_+$ to $u_-$ in the solution of **IVP** for $\delta > -1$. We recall that the case when $\delta = -1$ has been considered in (I), the large-time attractor in this case is of error function type.

# 2 Asymptotic solution as $t \to \infty$

In this section we develop the asymptotic structure of the solution to **IVP** as $t \to \infty$. We will develop the large-time solution of **IVP** for each of the two distinct cases when $u_+ > u_-$ and when $u_+ < u_-$. We must first begin by examining the asymptotic structure of the solution of **IVP** as $t \to 0$.

## 2.1 Asymptotic solution as $t \to 0$

We first consider region I where $x = O(1)$ as $t \to 0$. The cases $\delta > 0$ and $-1 < \delta < 0$ need to be considered separately. Following [3] it is straightforward to establish that:

(i) When $\delta > 0$,

$$u(x,t) = u_0(x) + \sum_{i=1}^{\lceil\delta\rceil} \frac{u_0^{(2i)}(x)}{i!}\, t^i - \begin{cases} \left(\frac{u_0(x)u_0'(x)}{(\delta+1)} - \frac{u_0^{(2\delta+2)}(x)}{(\delta+1)!}\right) t^{(\delta+1)} + o\left(t^{(\delta+1)}\right), & \delta \in \mathbb{N}, \\ \frac{u_0(x)u_0'(x)}{(\delta+1)}\, t^{(\delta+1)} + o\left(t^{(\delta+1)}\right), & \delta \notin \mathbb{N}, \end{cases} \tag{2.1}$$

as $t \to 0$ with $x = O(1)$, where $\lceil\delta\rceil$ is the smallest integer greater or equal to $\delta$. Expansion (2.1) (with (1.4)) remains uniform for $x \gg 1$ as $t \to 0$ and for $(-x) \gg 1$ as $t \to 0$. This completes the asymptotic structure as $t \to 0$, with expansion (2.1) of region I providing a uniform approximation to the solution of **IVP** as $t \to 0$ with $x = O(1)$.

(ii) When $-1 < \delta < 0$,

$$u(x,t) = u_0(x) + \sum_{i=1}^{\lceil\frac{-\delta}{(\delta+1)}\rceil} T_i(x)\, t^{i(\delta+1)} + \begin{cases} u_0''(x)t + o(t), & \frac{-\delta}{(\delta+1)} \notin \mathbb{N}, \\ \left[u_0''(x) + T_{\frac{1}{(\delta+1)}}(x)\right] t + o(t), & \frac{-\delta}{(\delta+1)} \in \mathbb{N}, \end{cases} \tag{2.2}$$

as $t \to 0$ with $x = O(1)$, where the functions $T_i(x)$ can be determined in terms of $u_0(x)$ straightforwardly. For example, $T_1(x) = -\frac{u_0(x)u_0'(x)}{(\delta+1)}$ and $T_2(x) = \frac{\left[u_0^2(x)u_0'(x)\right]'}{2(\delta+1)^2}$. Expansion (2.2) (with (1.4)) remains uniform for $x \gg 1$ as $t \to 0$ and for $(-x) \gg 1$ as $t \to 0$. This completes the asymptotic structure as $t \to 0$, with expansion (2.2) of region I providing a uniform approximation to the solution of **IVP** as $t \to 0$ with $x = O(1)$.

## 2.2 Asymptotic solution as $|x| \to \infty$

We now investigate the structure of the solution to **IVP** as $|x| \to \infty$ for $t = O(1)$. We begin by developing the structure of the solution to **IVP** as $x \to \infty$ with $t = O(1)$. The form of (2.1) ($\delta > 0$) and (2.2) ($-1 < \delta < 0$) for $x \gg 1$ indicates that in this region, which we label as region II$^+$, we must expand as

$$u(x,t) = u_+ + \frac{f_0(t)}{x^\gamma} + \frac{f_1(t)}{x^{\gamma+1}} + \frac{f_2(t)}{x^r} + \frac{f_3(t)}{x^s} + o\left(\frac{1}{x^s}\right) \tag{2.3}$$

as $x \to \infty$ with $t = O(1)$. A balancing of terms requires that

$$r = \begin{cases} 2\gamma + 1, & 0 < \gamma \leq 1, \\ \gamma + 2, & \gamma > 1, \end{cases}$$

$$s = \begin{cases} \gamma + 2, & 0 < \gamma < 1, \\ 4, & \gamma = 1, \\ 2\gamma + 1, & \gamma > 1. \end{cases}$$

On substituting (2.3) into equation (1.1) and solving at each order in turn we find, after matching with (2.1) (when $\delta > 0$) or (2.2) (when $-1 < \delta < 0$) as $t \to 0$, that

$$f_0(t) = A_R, \quad f_1(t) = \frac{\gamma u_+ A_R}{(\delta+1)} t^{(\delta+1)},$$

$$f_2(t) = \begin{cases} \dfrac{\gamma A_R^2}{(\delta+1)} t^{(\delta+1)}, & 0 < \gamma < 1, \\ \dfrac{A_R^2}{(\delta+1)} t^{(\delta+1)} + 2A_R t, & \gamma = 1, \\ \gamma(\gamma+1) A_R t, & \gamma > 1. \end{cases}$$

The function $f_3(t)$ is only required in what follows in the case $\gamma > 1$ and for brevity we do not report $f_3(t)$ for $0 < \gamma < 1$ or $\gamma = 1$. We readily obtain that

$$f_3(t) = \frac{\gamma A_R^2}{(\delta+1)} t^{(\delta+1)}, \quad \gamma > 1.$$

Thus, the structure of (2.3) depends on the parameters $\delta$ and $\gamma$. We observe that expansion (2.3) *remains uniform for* $t \gg 1$ *provided* $x \gg \lambda(t)$, but becomes nonuniform when $x = O(\lambda(t))$ for $t \gg 1$, where

$$\lambda(t) = \begin{cases} t^{(\delta+1)}, & \delta > -\frac{1}{2}, \gamma > 0, \\ t^{-\delta}, & -1 < \delta \leq -\frac{1}{2},\ \gamma \geq 1, \\ t^{\frac{\delta}{(\gamma-1)}}, & -1 < \delta \leq -\frac{1}{2},\ 0 < \gamma < 1. \end{cases} \tag{2.4}$$

We conclude by developing the structure of the solution to **IVP** as $x \to -\infty$ with $t = O(1)$. The form of (2.1) ($\delta > 0$) and (2.2) ($-1 < \delta < 0$) for $(-x) \gg 1$ dictates that in this region, which we label as region II$^-$, we expand as

$$u(x,t) = u_- + \frac{\hat{f}_0(t)}{(-x)^\gamma} + \frac{\hat{f}_1(t)}{(-x)^{\gamma+1}} + \frac{\hat{f}_2(t)}{(-x)^r} + \frac{\hat{f}_3(t)}{(-x)^s} + o\left(\frac{1}{(-x)^s}\right) \tag{2.5}$$

as $x \to -\infty$ with $t = O(1)$, and where $r$ and $s$ are as given above. On substituting (2.5) into equation (1.1) and solving at each order in turn we find, after matching with (2.1) (when $\delta > 0$) or (2.2) (when $-1 < \delta < 0$) as $t \to 0$, that

$$\hat{f}_0(t) = A_L, \quad \hat{f}_1(t) = -\frac{\gamma u_- A_L}{(\delta+1)} t^{(\delta+1)},$$

$$\hat{f}_2(t) = \begin{cases} -\dfrac{\gamma A_L^2}{(\delta+1)}\, t^{(\delta+1)}, & 0<\gamma<1, \\ -\dfrac{A_L^2}{(\delta+1)}\, t^{(\delta+1)} + 2A_L\, t, & \gamma = 1, \\ \gamma(\gamma+1)A_L\, t, & \gamma>1, \end{cases}$$

and

$$\hat{f}_3(t) = -\frac{\gamma A_L^2}{(\delta+1)} t^{(\delta+1)}, \quad \gamma > 1.$$

Thus, the structure of (2.5) depends on the parameters $\delta$ and $\gamma$. We observe again that expansion (2.5) *remains uniform for* $t \gg 1$ *provided* $(-x) \gg \lambda(t)$, but becomes nonuniform when $(-x) = O(\lambda(t))$ for $t \gg 1$, with $\lambda(t)$ given by (2.4). This completes the asymptotic structure of **IVP** as $|x| \to \infty$ with $t = O(1)$.

## 2.3 Asymptotic solution as $t \to \infty$

There are a number of cases to consider, which we develop in turn.

### 2.3.1 $\delta > -\frac{1}{2}$ and $u_+ > u_-$

We now investigate the structure of **IVP** as $t \to \infty$ when $\delta > -\frac{1}{2}$ and $u_+ > u_-$. We recall from Section 2.2 that expansions (2.3) and (2.5) of regions II$^+$ ($x \to \infty, t = O(1)$) and II$^-$ ($x \to -\infty, t = O(1)$) respectively, continue to remain uniform provided $|x| \gg t^{(\delta+1)}$ as $t \to \infty$. However, as already noted, a nonuniformity develops when $|x| = O\left(t^{(\delta+1)}\right)$. We begin by considering the asymptotic structure as $t \to \infty$ moving in from region II$^+$, when $x \gg t^{(\delta+1)}$ as $t \to \infty$. To proceed we introduce a new region labelled as region III$^+$, in which $x = O\left(t^{(\delta+1)}\right)$ as $t \to \infty$. To examine region III$^+$ we introduce the scaled coordinate

$$y = \frac{x}{t^{(\delta+1)}}, \tag{2.6}$$

where $y = O(1)$ as $t \to \infty$ in region III$^+$, whilst the form of expansion (2.3) in region II$^+$, when $t \gg 1$ and $x = O\left(t^{(\delta+1)}\right)$ requires that we expand as

$$u(y,t) = u_+ + g_0(y)t^{-\gamma(\delta+1)} + o\left(t^{-\gamma(\delta+1)}\right) \tag{2.7}$$

as $t \to \infty$ with $y = O(1)$, and recalling that $\delta > -\frac{1}{2}$. After substituting (2.7) into equation (1.1) (when written in terms of $y$ and $t$), the leading order problem for $g_0(y)$ becomes

$$\left(y - \frac{u_+}{(\delta+1)}\right) g_0' + \gamma g_0 = 0, \quad y = O(1), \tag{2.8}$$

$$g_0(y) \sim \frac{A_R}{y^\gamma} + \frac{\gamma u_+ A_R}{(\delta+1)y^{(\gamma+1)}} \quad \text{as} \quad y \to \infty, \tag{2.9}$$

with condition (2.9) arising from matching expansion (2.7) ($y \gg 1$) with the far field expansion (2.3) ($x = O\left(t^{(\delta+1)}\right)$). The solution of (2.8), (2.9) is readily obtained as

$$g_0(y) = A_R \left(y - \frac{u_+}{(\delta+1)}\right)^{-\gamma}, \quad y > \frac{u_+}{(\delta+1)}. \tag{2.10}$$

An examination of (2.10) reveals that $g_0(y)$ develops a singularity as $y \to \left(\frac{u_+}{(\delta+1)}\right)^+$, and thus expansion (2.7) becomes nonuniform when $y = \frac{u_+}{(\delta+1)} + o(1)$ as $t \to \infty$ and region III$^+$ is restricted to $y = \frac{u_+}{(\delta+1)} + O(1)$ as $t \to \infty$. To proceed we examine this region of nonuniformity by introducing the new region, labelled as region IV$^+$. In region IV$^+$, we write

$$y = \frac{u_+}{(\delta+1)} + \xi t^{-r} \tag{2.11}$$

with $\xi = O(1)$ as $t \to \infty$, and $r > 0$ to be determined, whilst we expand in region IV$^+$ in the form

$$u(\xi,t) = u_+ + F(\xi)t^{-s} + o\left(t^{-s}\right) \tag{2.12}$$

as $t \to \infty$ with $\xi = O(1)$, and $s > 0$ is to be determined. On substitution of (2.11) and (2.12) into equation (1.1) (when written in terms of $\xi$ and $t$) we obtain

$$-sF + FF_\xi t^{(r-s)} + (r-\delta-1)\,\xi F_\xi = F_{\xi\xi} t^{(2r-2\delta-1)}. \tag{2.13}$$

To obtain the most structured leading order balance in (2.13) we require that

$$s = r \tag{2.14}$$

and then that

$$r = \begin{cases} \frac{\gamma(\delta+1)}{(\gamma+1)}, & 0 < \gamma < 2\delta+1, \\ \left(\delta+\frac{1}{2}\right), & \gamma \geq 2\delta+1. \end{cases} \tag{2.15}$$

We consider the cases $0 < \gamma < 2\delta+1$, $\gamma > 2\delta+1$ and $\gamma = 2\delta+1$ separately.

(i) $0 < \gamma < 2\delta+1$

In this case the leading order problem is

$$\left(F - \frac{(\delta+1)}{(\gamma+1)}\xi\right) F_\xi - \frac{\gamma(\delta+1)}{(\gamma+1)} F = 0, \quad -\infty < \xi < \infty. \tag{2.16}$$

The matching condition with region III$^+$ requires

$$F(\xi) \sim A_R \xi^{-\gamma} \quad \text{as} \quad \xi \to \infty, \tag{2.17}$$

We note that equation (2.16) admits the exact solution

$$F(\xi) = (1+\delta)\xi, \quad -\infty < \xi < \infty.$$

In general, equation (2.16) is of homogeneous type, and admits a quadrature, after which the solution to (2.16) with (2.17) is given implicity by,

$$\xi = \left(\frac{A_R}{F(\xi)}\right)^{\frac{1}{\gamma}} + \frac{F(\xi)}{(\delta+1)}, \quad -\infty < \xi < \infty. \tag{2.18}$$

It follows from (2.18) that (on recalling that $A_R < 0$),

$$F(\xi) < 0 \quad \text{for all} \quad -\infty < \xi < \infty, \tag{2.19}$$

$$F(\xi) \quad \text{is strictly monotone increasing, with} \quad -\infty < \xi < \infty, \tag{2.20}$$

$$F(\xi) \sim A_R \xi^{-\gamma} + \frac{A_R^2 \gamma}{(\delta+1)} \xi^{-(2\gamma+1)} \quad \text{as} \quad \xi \to \infty, \tag{2.21}$$

$$F(\xi) \sim (\delta+1)\xi - (-A_R)^{\frac{1}{\gamma}} (\delta+1)^{\left(1-\frac{1}{\gamma}\right)} (-\xi)^{-\frac{1}{\gamma}} \quad \text{as} \quad \xi \to -\infty. \tag{2.22}$$

(ii) $\gamma > 2\delta+1$

In this case the leading order problem is

$$F_{\xi\xi} - \left(F - \frac{1}{2}\xi\right) F_\xi + \left(\delta + \frac{1}{2}\right) F = 0, \quad -\infty < \xi < \infty. \tag{2.23}$$

The principal matching condition with region III$^+$ requires

$$F(\xi) \to 0 \quad \text{as} \quad \xi \to \infty. \tag{2.24}$$

Solutions to (2.23) which satisfy (2.24) have the form

$$F(\xi) \sim A\xi^{-(2\delta+1)} + B\xi^{2\delta}e^{-\frac{1}{4}\xi^2} \tag{2.25}$$

as $\xi \to \infty$, with $A$ and $B$ being arbitrary constants. We observe immediately, that the algebraic power in (2.25) will preclude matching at two-terms with region III$^+$. A local passive adjustment region is required. This is achieved by requiring $A = 0$ in (2.25), and the passive region is then located at $\xi = \xi_p(t) + O(1)$ as $t \to \infty$, where,

$$\xi_p^2(t) = 2\left(\gamma - (2\delta+1)\right)\ln t + O\big(\ln\big((\gamma - (2\delta+1))\ln t\big)\big), \tag{2.26}$$

as $t \to \infty$. The details of the passive region are omitted for brevity. Thus the leading order problem is now (2.23) with

$$F(\xi) \sim B\xi^{2\delta}e^{-\frac{1}{4}\xi^2} \quad \text{as} \quad \xi \to \infty \tag{2.27}$$

for some constant $B$. We again note that equation (2.23) admits the exact solution

$$F(\xi) = (1+\delta)\xi, \quad -\infty < \xi < \infty,$$

and for region IV$^+$ to act as a transition region, we require the solution of (2.23) and (2.27) to satisfy

$$F(\xi) \sim (1+\delta)\xi \quad \text{as} \quad \xi \to -\infty. \tag{2.28}$$

An examination of equation (2.23), with condition (2.25), reveals that for each *fixed* $A \leq 0$, the $(F, F')$ phase portrait of solutions to (2.23) takes the form of Figure 1(a) when $A < 0$ and Figure 1(b) when $A = 0$. The local structure of the equilibrium point $(0,0)$ is obtained via a linearization of equation (2.23) (note that although (2.23) is non-autonomous, local phase paths do not intersect since the general form (2.25) is restricted by either fixing $A < 0$ or setting $A = 0$, after which local phase paths are a one parameter family, parameterized by $B \in \mathbb{R}$). The straightline phase path follows from the exact solution $F(\xi) = (1+\delta)\xi$, with local phase paths to this line again determined by a linearization of (2.23) about this exact solution. The remaining phase paths, and, in particular, the existence of the phase path $\mathcal{H}^*$, are deduced by continuous deformation. In addition, the structure of the phase portrait has been confirmed numerically. Each phase path connecting to $(0,0)$ corresponds to a unique $B \in \mathbb{R}$. An examination of Figure 1 then enables us to conclude that there exists a unique $B^*$, which corresponds to the phase path $\mathcal{H}^*$, for which (2.23) has a solution $F = F^*(\xi)$, $-\infty < \xi < \infty$, which satisfies conditions (2.27) and (2.28). It is readily established that $B^* < 0$, whilst,

$$F^*(\xi) < 0 \quad \forall \quad \xi \in \mathbb{R},$$

$$F^{*'}(\xi) > 0 \quad \forall \quad \xi \in \mathbb{R} \tag{2.29}$$

$$F^*(\xi) \sim (1+\delta)\xi - C^*(-\xi)^{-\frac{1}{(2\delta+1)}} \quad \text{as} \quad \xi \to -\infty, \tag{2.30}$$

for some constant $C^* > 0$. A numerical study of initial-value problem (2.23) and (2.27) using a shooting method again reveals that there exists a value $B = B^* < 0$ such that boundary condition (2.30) is satisfied. A numerical determination of $F = F^*(\xi)$ when $\delta = 1$ is given in Figure 2, for which $B^*$ and $C^*$ are determined as $B^* = -1.95...$ and $C^* = 5.05....$

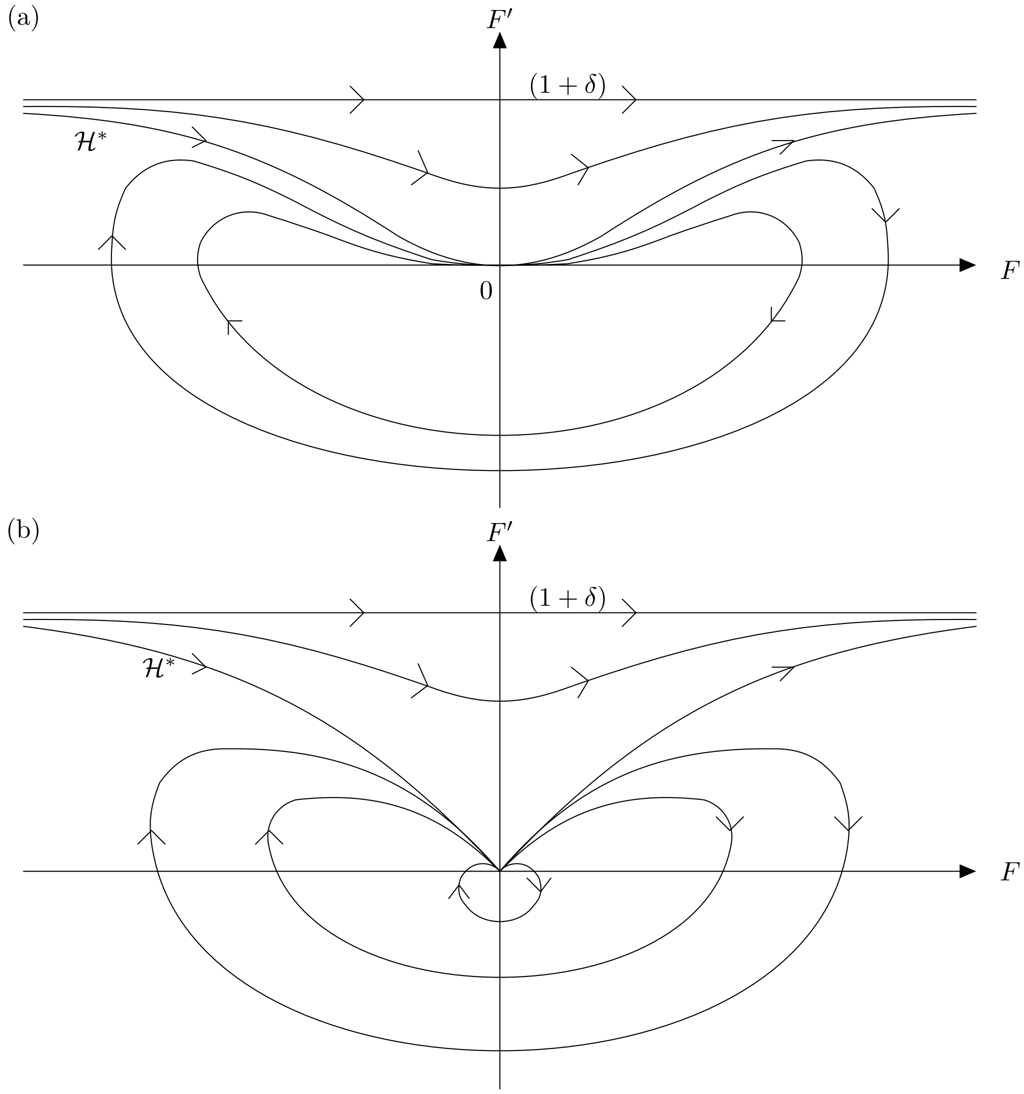


Figure 1: The $(F, F')$ phase plane for (2.23) with fixed $A$. (a) has $A < 0$ and (b) has $A = 0$.

(iii) $\gamma = 2\delta + 1$

In this case the leading order problem is again equation (2.23), but now matching with region III$^+$ is possible with the matching condition requiring,

$$F(\xi) \sim A_R \xi^{-(2\delta+1)} \quad \text{as} \quad \xi \to \infty. \tag{2.31}$$

Recalling that $A_R < 0$, then it follows from Figure 1(a), on taking $A = A_R$, that there exists $B = B^*(A_R)$, such that (2.23) and (2.31) has a solution $F = F_R^*(\xi)$, $-\infty < \xi < \infty$, corresponding to the phase path $\mathcal{H}^*$ in Figure 1(a). In particular,

$$F_R^*(\xi) < 0 \quad \forall \quad \xi \in \mathbb{R},$$

$$F_R^{*'}(\xi) > 0 \quad \forall \quad \xi \in \mathbb{R} \tag{2.32}$$

$$F_R^*(\xi) \sim A_R \xi^{-(2\delta+1)} + B^*(A_R)\xi^{2\delta} e^{-\frac{1}{4}\xi^2} \quad \text{as} \quad \xi \to \infty, \tag{2.33}$$

$$F_R^*(\xi) \sim (1+\delta)\xi - C^*(A_R)(-\xi)^{-\frac{1}{(2\delta+1)}} \quad \text{as} \quad \xi \to -\infty,$$

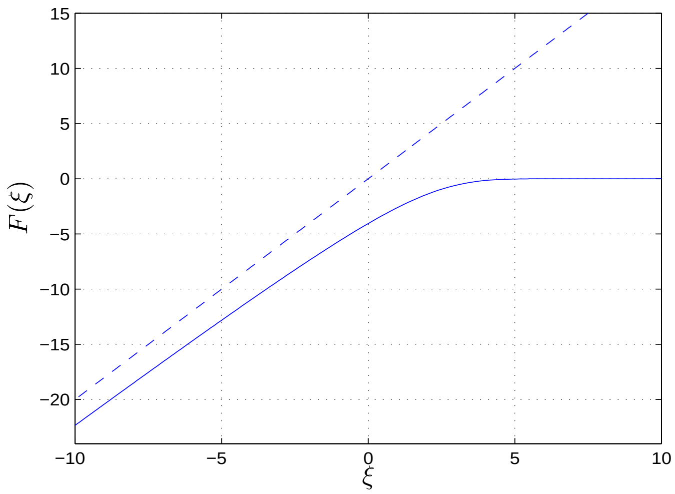


Figure 2: A graph of $F(\xi)$ against $\xi$ when $\delta = 1$. The solid line represents the solution $F^*(\xi)$ while the dashed line represents the solution $F(\xi) = (\delta + 1)\xi$.

for some constant $C^*(A_R) > 0$.

As $\xi \to -\infty$, we move out of the localized region IV$^+$ into region EW where now $y = \frac{u_+}{(\delta+1)} - O(1)$. It is convenient at this stage to leave region EW until later, and to next develop the asymptotic structure of $u(y,t)$ as $t \to \infty$, moving in from region II$^-$ (when $(-y) \gg 1$) to $y = O(1)$ as $t \to \infty$. To proceed we introduce a new region, labelled as region III$^-$. The details of this region follow those given for region III$^+$ and are not repeated here. In region III$^-$ we have that

$$u(y,t) = u_- + A_L \left( \frac{u_-}{(\delta+1)} - y \right)^{-\gamma} t^{-\gamma(\delta+1)} + o\left(t^{-\gamma(\delta+1)}\right) \tag{2.34}$$

as $t \to \infty$ with $y = \frac{u_-}{(\delta+1)} - O(1)$. Expansion (2.34) becomes nonuniform when

$$y = \frac{u_-}{(\delta+1)} + O\left(t^{-r}\right)$$

as $t \to \infty$ with $r > 0$ as given in (2.15). Thus, we examine this region of nonuniformity by introducing region IV$^-$. We introduce the scaled coordinate $\xi$, via

$$y = \frac{u_-}{(\delta+1)} + \xi t^{-r}$$

with $\xi = O(1)$ as $t \to \infty$, and expansion (2.34) dictating that we expand as

$$u(\xi,t) = u_- + \hat{F}(\xi)t^{-r} + o\left(t^{-r}\right) \tag{2.35}$$

as $t \to \infty$ with $\xi = O(1)$. The details are identical to those for region IV$^+$, and are not repeated here. In fact, making the replacement of $A_R$ by $(-A_L)$, we obtain for $\hat{F}(\xi)$, $-\infty < \xi < \infty$, that

$$\hat{F}(\xi) = \begin{cases} -F(-\xi), & 0 < \gamma < 2\delta + 1, \\ -F^*(-\xi), & \gamma > 2\delta + 1, \\ -F^*_R(-\xi), & \gamma = 2\delta + 1. \end{cases}$$

As $\xi \to \infty$, we move out of the localized region IV$^-$ into region EW where now $y = \frac{u_-}{(\delta+1)} + O(1)$. To complete the asymptotic structure as $t \to \infty$, it remains to consider region EW, where $\frac{u_-}{(\delta+1)} + O(1) \leq y \leq \frac{u_+}{(\delta+1)} - O(1)$ as $t \to \infty$. Expansions (2.12) and (2.35) of regions IV$^+$ and IV$^-$ respectively dictate that we expand in region EW as,

$$u(y,t) = G_0(y) + G_1(y)t^{-(\delta+1)} + G_2(y)t^{-2(\delta+1)} + o\left(t^{-2(\delta+1)}\right) \tag{2.36}$$

as $t \to \infty$ with $\frac{u_-}{(\delta+1)} + O(1) \leq y \leq \frac{u_+}{(\delta+1)} - O(1)$. On substitution of (2.36) into equation (1.1) (when written in terms of $y$ and $t$) we obtain at leading order that

$$G_0'\Big(G_0 - (\delta+1)y\Big) = 0, \quad \frac{u_-}{(\delta+1)} < y < \frac{u_+}{(\delta+1)}. \tag{2.37}$$

Equation (2.37) is to be solved subject to the matching conditions with region IV$^+$ and region IV$^-$, namely,

$$G_0(y) \sim u_+ + (\delta+1)\left(y - \frac{u_+}{(\delta+1)}\right) \quad \text{as} \quad y \to \left(\frac{u_+}{(\delta+1)}\right)^-, \tag{2.38}$$

and

$$G_0(y) \sim u_- + (\delta+1)\left(y - \frac{u_-}{(\delta+1)}\right) \quad \text{as} \quad y \to \left(\frac{u_-}{(\delta+1)}\right)^+. \tag{2.39}$$

The solution to (2.37) the subject to (2.38) and (2.39) is readily obtained as

$$G_0(y) = (\delta+1)y, \quad \frac{u_-}{(\delta+1)} < y < \frac{u_+}{(\delta+1)}. \tag{2.40}$$

The function $G_1(y)$ remains undetermined, being a remnant of the global evolution when $t = O(1)$. However, matching to region IV$^\pm$ requires

$$G_1(y) \sim \begin{cases} A_L^{\frac{1}{\gamma}}(\delta+1)^{\left(1-\frac{1}{\gamma}\right)}\left(y - \frac{u_-}{(\delta+1)}\right)^{-\frac{1}{\gamma}} & \text{as} \quad y \to \left(\frac{u_-}{(\delta+1)}\right)^+, \\ -(-A_R)^{\frac{1}{\gamma}}(\delta+1)^{\left(1-\frac{1}{\gamma}\right)}\left(\frac{u_+}{(\delta+1)} - y\right)^{-\frac{1}{\gamma}} & \text{as} \quad y \to \left(\frac{u_+}{(\delta+1)}\right)^-, \end{cases} \tag{2.41}$$

when $0 < \gamma < (2\delta+1)$, whilst

$$G_1(y) \sim \begin{cases} C^*\left(y - \frac{u_-}{(\delta+1)}\right)^{-\frac{1}{(2\delta+1)}} & \text{as} \quad y \to \left(\frac{u_-}{(\delta+1)}\right)^+, \\ -C^*\left(\frac{u_+}{(\delta+1)} - y\right)^{-\frac{1}{(2\delta+1)}} & \text{as} \quad y \to \left(\frac{u_+}{(\delta+1)}\right)^-, \end{cases} \tag{2.42}$$

when $\gamma > 2\delta+1$, and,

$$G_1(y) \sim \begin{cases} C^*(-A_L)\left(y - \frac{u_-}{(\delta+1)}\right)^{-\frac{1}{(2\delta+1)}} & \text{as} \quad y \to \left(\frac{u_-}{(\delta+1)}\right)^+, \\ -C^*(A_R)\left(\frac{u_+}{(\delta+1)} - y\right)^{-\frac{1}{(2\delta+1)}} & \text{as} \quad y \to \left(\frac{u_+}{(\delta+1)}\right)^-, \end{cases} \tag{2.43}$$

when $\gamma = 2\delta+1$. At next order, we obtain

$$G_2(y) = \frac{1}{(\delta+1)}G_1(y)G_1'(y).$$

The asymptotic expansion in region EW is then, via (2.36), given by

$$u(y,t) = (\delta+1)y + G_1(y)t^{-(\delta+1)} + \frac{1}{(\delta+1)}G_1(y)G_1'(y)t^{-2(\delta+1)} + o\left(t^{-2(\delta+1)}\right) \tag{2.44}$$

as $t \to \infty$ with $\frac{u_-}{(\delta+1)} + O(1) \leq y \leq \frac{u_+}{(\delta+1)} - O(1)$.

The asymptotic structure of **IVP** as $t \to \infty$ when $u_+ > u_-$ is now complete. A uniform approximation has been given through regions II$^\pm$, III$^\pm$, IV$^\pm$ and EW. A

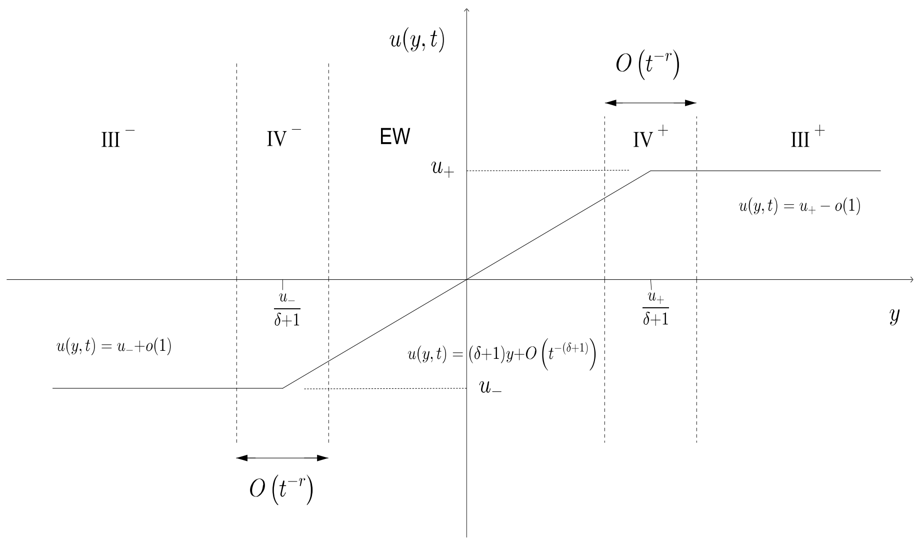


Figure 3: A schematic representation of the asymptotic structure of $u(y,t)$ in the $(y,u)$ plane as $t \to \infty$ for **IVP** when $u_+ > u_-$.

schematic representation of the location and thickness of the asymptotic regions as $t \to \infty$ is given for in Figure 3 (we recall that depending on the problem parameters there may be passive regions located at $\frac{u_+}{(\delta+1)}$ and $\frac{u_-}{(\delta+1)}$ which allow for the reordering of terms in the expansions in regions III$^+$ and III$^-$ respectively). The large-t attractor for the solution of **IVP** when $u_+ > u_-$ is the expansion wave which allows for the adjustment of the solution from $u_+$ to $u_-$. We can summarize the results of this section in the following theorem. It is first convenient to introduce the function $u_E : \mathbb{R} \to \mathbb{R}$ such that

$$u_E(\lambda) = \begin{cases} u_+, & \lambda > \frac{u_+}{(\delta+1)}, \\ (\delta+1)\lambda, & \frac{u_+}{(\delta+1)} \leq \lambda \leq \frac{u_-}{(\delta+1)}, \\ u_-, & \lambda < \frac{u_-}{(\delta+1)}. \end{cases}$$

We then have,

**Proposition 1.** *Let $u : \mathbb{R} \times [0,\infty) \to \mathbb{R}$ be the solution to* **IVP** *when $\gamma > 0$, $\delta > -\frac{1}{2}$ and $u_+ > u_-$. In terms of the coordinate $y = \frac{x}{t^{(\delta+1)}}$, on writing,*

$$u(y,t) = u_E(y) + R(y,t)$$

*for $(y,t) \in \mathbb{R} \times [0,\infty)$, then $R(y,t) \to 0$ as $t \to \infty$ uniformly for $y \in \mathbb{R}$. In particular,*

$$R(y,t) = \begin{cases} O\left(\frac{t^{-\gamma(\delta+1)}}{(1+|y|^{\gamma})}\right) & \text{in regions } III^{\pm}, \\ O\left(t^{-r}\right) & \text{in regions } IV^{\pm}, \\ O\left(t^{-(\delta+1)}\right) & \text{in region } EW, \end{cases}$$

*as $t \to \infty$, with $r$ given in (2.15).*

### 2.3.2 $\delta > -\frac{1}{2}$ and $u_+ < u_-$

We now investigate the structure of **IVP** as $t \to \infty$ when $\delta > -\frac{1}{2}$ and $u_+ < u_-$. We recall from Section 2.2 that expansions (2.3) and (2.5) of regions II$^+$ ($x \to \infty, t = O(1)$) and II$^-$ ($x \to -\infty, t = O(1)$) respectively, continue to remain uniform provided $|x| \gg t^{(\delta+1)}$ as $t \to \infty$. However, as already noted a nonuniformity develops when

$|x| = O\left(t^{(\delta+1)}\right)$. As in Section 2.3.1, we introduce the scaled coordinate

$$y = \frac{x}{t^{(\delta+1)}}, \tag{2.45}$$

where $y = O(1)$, and begin by summarizing the asymptotic structure as $t \to \infty$ in regions III$^+$ and III$^-$, the details of which follow, after minor modification, those given in Section 2.3.1 and are not repeated here.

Region III$^+$

$$u(y,t) = u_+ + A_R\left(y - \frac{u_+}{(\delta+1)}\right)^{-\gamma} t^{-\gamma(\delta+1)} + o\left(t^{-\gamma(\delta+1)}\right) \tag{2.46}$$

as $t \to \infty$ with $y = \frac{u_+}{(\delta+1)} + O(1)$, and where $A_R > 0$.

Region III$^-$

$$u(y,t) = u_- + A_L\left(\frac{u_-}{(\delta+1)} - y\right)^{-\gamma} t^{-\gamma(\delta+1)} + o\left(t^{-\gamma(\delta+1)}\right) \tag{2.47}$$

as $t \to \infty$ with $y = \frac{u_-}{(\delta+1)} - O(1)$, and where $A_L < 0$.

In this case, when $u_+ < u_-$, expansions (2.46) and (2.47) must become nonuniform when $y = \alpha + o(1)$, where $\alpha \in \left(\frac{u_+}{(\delta+1)}, \frac{u_-}{(\delta+1)}\right)$ and is to be determined. To examine this region, which we label as region SS, we introduce the scaled coordinate

$$z = (y - \alpha)\psi(t)^{-1} = O(1), \tag{2.48}$$

where $\psi(t) = o(1)$ as $t \to \infty$, is an as yet undetermined gauge function, and expand in the form

$$u(z,t) = U(z) + o(1) \tag{2.49}$$

as $t \to \infty$ with $z = O(1)$. On substituting (2.49) into equation (1.1) (when written in terms of $z$ and $t$) we find that to obtain the most structured leading order balance we require, without loss of generality,

$$\psi(t) = t^{-(2\delta+1)}. \tag{2.50}$$

At leading order we then obtain

$$U_{zz} - UU_z + \alpha(\delta+1)U_z = 0, \quad -\infty < z < \infty. \tag{2.51}$$

On integrating (2.51) we obtain

$$U_z = \frac{U^2}{2} - \alpha(\delta+1)U + C, \quad -\infty < z < \infty, \tag{2.52}$$

where $C$ is a constant of integration. Equation (2.52) is to be solved subject to the leading order matching conditions with regions III$^\pm$, namely,

$$U(z) \to \begin{cases} u_+ & \text{as} \quad z \to \infty, \\ u_- & \text{as} \quad z \to -\infty. \end{cases} \tag{2.53}$$

The solution to (2.52) subject to boundary conditions (2.53) requires that

$$\alpha = \frac{(u_+ + u_-)}{2(\delta+1)}, \qquad C = \frac{u_+u_-}{2},$$

and is given by the Taylor shock profile (see [8])

$$U(z) = \frac{(u_+ + u_-)}{2} - \frac{(u_- - u_+)}{2}\tanh\left(\frac{(u_- - u_+)}{4}z + \phi_c\right), \quad -\infty < z < \infty, \tag{2.54}$$

where $\phi_c$ is a globally determined constant. We note that

$$U(z) \sim \begin{cases} u_+ + (u_- - u_+)\exp\left(-\frac{(u_- - u_+)}{2}z - \phi_c\right) & \text{as} \quad z \to \infty, \\ u_- - (u_- - u_+)\exp\left(\frac{(u_- - u_+)}{2}z + \phi_c\right) & \text{as} \quad z \to -\infty. \end{cases} \tag{2.55}$$

The similarity solution (2.54) represents a wavefront connecting $u_+$ (as $z \to \infty$) to $u_-$ (as $z \to -\infty$). Specifically, when $\delta > 0$ the Taylor shock profile is located $x = \alpha t^{(\delta+1)}$ and contained in a localized region of thickness $O(t^{-\delta})$ as $t \to \infty$ (the profile steepens as $t \to \infty$). The Taylor shock is accelerating in the $+x$ $(-x)$ direction as $t \to \infty$ when $-u_- < u_+ < u_-$ $(u_+ < u_- < -u_+)$ respectively. When $-\frac{1}{2} < \delta < 0$ the Taylor shock profile is located at $x = \alpha t^{(\delta+1)}$ and contained within a region of thickness $O(t^{|\delta|})$ as $t \to \infty$ (the profile becomes stretched as $t \to \infty$). The Taylor shock is decelerating in the $+x$ $(-x)$ direction as $t \to \infty$ when $-u_- < u_+ < u_-$ $(u_+ < u_- < -u_+)$ respectively.

However, we readily observe that matching expansion (2.49) (as $z \to \infty$) to expansion (2.46) (as $y \to \frac{(u_+ + u_-)}{2(\delta+1)}^+$) at next order fails and we require a transition region, which we label TR$^+$. To examine region TR$^+$ we introduce the scaled coordinate $\eta$ by

$$y = \frac{(u_+ + u_-)}{2(\delta+1)} + \frac{2\gamma(\delta+1)}{(u_- - u_+)}\frac{\ln t}{t^{(2\delta+1)}} + \eta t^{-(2\delta+1)}, \tag{2.56}$$

with $\eta = O(1)$ as $t \to \infty$ in region TR$^+$ (that is, $z = \frac{2\gamma(\delta+1)}{(u_- - u_+)}\ln t + \eta$). The $t^{-(2\delta+1)}\ln t$ shift in (2.56) is dictated by the matching requirements with regions III$^+$ and SS, as will be seen. The form of expansion (2.54) (for $z \gg 1$) then suggests that in region TR$^+$ we expand as

$$u(\eta, t) = u_+ + F(\eta)t^{-\gamma(\delta+1)} + +o\left(t^{-\gamma(\delta+1)}\right) \tag{2.57}$$

as $t \to \infty$ with $\eta = O(1)$. On substitution of (2.57) into equation (1.1) (when written in terms of $\eta$ and $t$) we obtain at leading order that

$$F_{\eta\eta} + \frac{(u_- - u_+)}{2}F_\eta = 0, \quad -\infty < \eta < \infty. \tag{2.58}$$

Equation (2.58) is to be solved subject to the following matching conditions with region III$^+$ (as $\eta \to \infty$) and region SS (as $\eta \to -\infty$), namely,

$$F(\eta) \sim \begin{cases} A_R\left(\frac{2(\delta+1)}{(u_- - u_+)}\right)^\gamma & \text{as} \quad \eta \to \infty, \\ (u_- - u_+)e^{-\phi_c}e^{-\frac{(u_- - u_+)}{2}\eta} & \text{as} \quad \eta \to -\infty. \end{cases} \tag{2.59}$$

The solution to (2.58), (2.59) is readily obtained as

$$F(\eta) = A_R\left(\frac{2(\delta+1)}{(u_- - u_+)}\right)^\gamma + (u_- - u_+)e^{-\phi_c}e^{-\frac{(u_- - u_+)}{2}\eta}, \quad -\infty < \eta < \infty. \tag{2.60}$$

Therefore, we have in region TR$^+$ that

$$u(\eta, t) = u_+ + \left(A_R\left(\frac{2(\delta+1)}{(u_- - u_+)}\right)^\gamma + (u_- - u_+)e^{-\phi_c}e^{-\frac{(u_- - u_+)}{2}\eta}\right)t^{-\gamma(\delta+1)} + o\left(t^{-\gamma(\delta+1)}\right) \tag{2.61}$$

as $t \to \infty$ with $\eta = O(1)$. We also note that matching between region TR$^+$ and region SS, via (2.61) and (2.49), also determines that the correction term in (2.49) must be $O(t^{-\gamma(\delta+1)})$.

Finally, we conclude this case by noting that matching expansion (2.49) (as $z \to -\infty$) to expansion (2.47) (as $y \to \frac{(u_+ + u_-)}{2(\delta+1)}^-$) similarly fails and we require a corresponding transition region, which we label TR$^-$. To examine region TR$^-$ we introduce the scaled coordinate $\hat{\eta}$ by

$$y = \frac{(u_+ + u_-)}{2(\delta+1)} - \frac{2\gamma(\delta+1)}{(u_- - u_+)} \frac{\ln t}{t^{(2\delta+1)}} + \hat{\eta} t^{-(2\delta+1)}, \tag{2.62}$$

so that $\hat{\eta} = O(1)$ as $t \to \infty$ in region TR$^-$. The details of region TR$^-$ follow, after minor modification, those given for region TR$^+$ above and are are not repeated here. In summary we have in region TR$^-$ that

$$u(\hat{\eta}, t) = u_- + \left( A_L \left( \frac{2(\delta+1)}{u_- - u_+} \right)^{\gamma} - (u_- - u_+) e^{\phi_c} e^{-\frac{(u_- - u_+)}{2}\hat{\eta}} \right) t^{-\gamma(\delta+1)} + o\left( t^{-\gamma(\delta+1)} \right) \tag{2.63}$$

as $t \to \infty$ with $\hat{\eta} = O(1)$.

The asymptotic structure of the solution of **IVP** as $t \to \infty$ when $\delta > -\frac{1}{2}$ and $u_+ < u_-$ is now complete. A uniform approximation has been given through regions II$^{\pm}$, III$^{\pm}$, TR$^{\pm}$ and SS. A schematic representation of the location and thickness of the asymptotic regions as $t \to \infty$ is given in Figure 4. The large-t attractor for the solution of **IVP** when $u_+ < u_-$ is the Taylor shock profile which allows for the adjustment of the solution from $u_+$ to $u_-$. The above asymptotic structure can be summarized in the following theorem. It is first convenient to introduce the function $u_T : \mathbb{R} \to \mathbb{R}$ such that

$$u_T(\lambda) = \frac{(u_+ + u_-)}{2} - \frac{(u_- - u_+)}{2} \tanh\left( \frac{(u_- - u_+)\lambda}{4} \right) \quad \forall \lambda \in \mathbb{R}.$$

We then have,

**Proposition 2.** *Let $u : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ be the solution to* **IVP** *when $\gamma > 0$, $\delta > -\frac{1}{2}$ and $u_+ < u_-$. In terms of the coordinate $y = xt^{-(\delta+1)}$, there exists a globally determined constant $\phi$, such that, on writing*

$$u(y, t) = u_T \left( \left( y - \frac{(u_+ + u_-)}{2(\delta+1)} \right) t^{(2\delta+1)} + \phi \right) + R(y, t)$$

*for $(y, t) \in \mathbb{R} \times [0, \infty)$, then $R(y, t) \to 0$ as $t \to \infty$ uniformly for $y \in \mathbb{R}$, with*

$$R(y, t) = O\left( \frac{t^{-\gamma(\delta+1)}}{(1 + |y|^{\gamma})} \right)$$

*as $t \to \infty$, uniformly for $y \in \mathbb{R}$.*

We remark that the globally determined constant $\phi$ is a consequence of the evolution over all $t > 0$, and as we have seen, is indeterminate through our asymptotic analysis as $t \to \infty$.

### 2.3.3 $-1 < \delta < -\frac{1}{2}$

We recall from Section 2.2 that in this case we need to consider the cases $\gamma \geq 1$ and $0 < \gamma < 1$ separately. We begin by considering the case $\gamma \geq 1$.

(a) $\boldsymbol{\gamma \geq 1}$

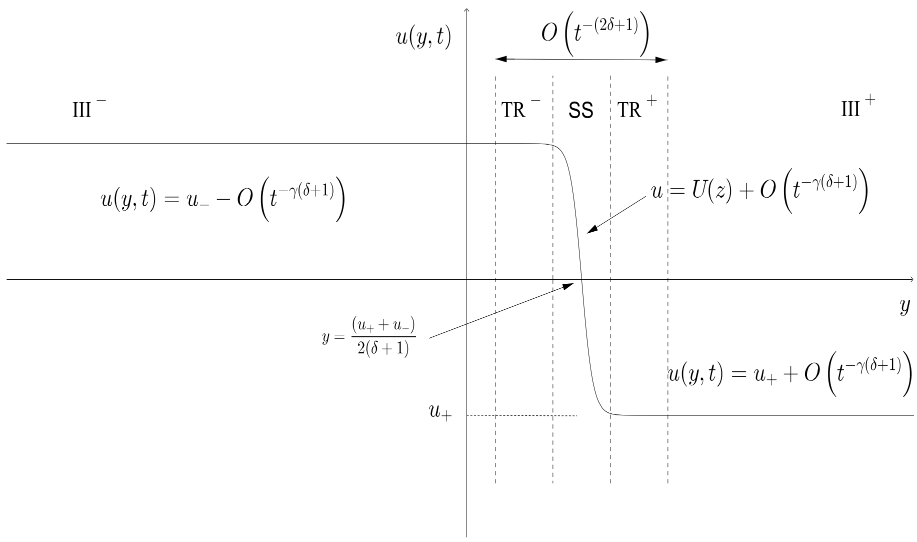


Figure 4: A schematic representation of the asymptotic structure of $u(y,t)$ in the $(y,u)$ plane as $t \to \infty$ for **IVP** when $u_+ < u_-$. We recall that region SS is located at $y = \frac{u_+ + u_-}{2(\delta+1)}$ (with thickness $O\left(t^{-(2\delta+1)}\right)$ as $t \to \infty$), while regions TR$^\pm$ are located at $y = \frac{u_+ + u_-}{2(\delta+1)} \pm \frac{2\gamma(\delta+1)}{u_- - u_+} \frac{\ln t}{t^{2\delta+1}}$ (with thickness $O\left(t^{-(2\delta+1)}\right)$) as $t \to \infty$.

In this case expansions (2.3) and (2.5) of regions II$^+$ ($x \to \infty$, $t = O(1)$) and II$^-$ ($x \to -\infty$, $t = O(1)$) respectively , continue to remain uniform provided $|x| \gg t^{-\delta}$. However, as already noted a nonuniformity develops when $|x| = O(t^{-\delta})$. We begin by considering the asymptotic structure as $t \to \infty$ by moving in from region II$^+$ when $x \gg t^{-\delta}$. To proceed we introduce a new region, labelled as region III$^+$. To examine region III$^+$ we introduce the scaled coordinate

$$y = x\, t^{\delta}, \tag{2.64}$$

where $y = O(1)$ as $t \to \infty$, and look for an expansion of the form

$$u(y,t) = u_+ + F_0(y) t^{\gamma\delta} + F_1(y) t^{(\gamma\delta+2\delta+1)} + o\left(t^{(\gamma\delta+2\delta+1)}\right) \tag{2.65}$$

as $t \to \infty$ with $y = O(1)$. On substituting (2.65) into equation (1.1) (when written in terms of $y$ and $t$) and solving at each order in turn, we find (after matching with (2.3)) that

$$u(y,t) = u_+ + A_R y^{-\gamma} t^{\gamma\delta} + \left(\frac{u_+ A_R \gamma}{(\delta+1)} y^{-(\gamma+1)} + \gamma(\gamma+1) A_R y^{-(\gamma+2)}\right) t^{(\gamma\delta+2\delta+1)} + o\left(t^{(\gamma\delta+2\delta+1)}\right) \tag{2.66}$$

as $t \to \infty$ with $y = O(1)$. Expansion (2.66) becomes nonuniform when $y = O(t^{(\delta+\frac{1}{2})})$ as $t \to \infty$ (that is, when $x = O(t^{\frac{1}{2}})$ as $t \to \infty$). We next consider the asymptotic structure as $t \to \infty$ by moving in from region II$^-$ when $(-x) \gg t^{\delta}$. To proceed we introduce a new region, labelled as region III$^-$. The details of this region follow, after minor modification, those given for region III$^+$ above. We have in region III$^-$ that

$$u(y,t) = u_- + A_L (-y)^{-\gamma} t^{\gamma\delta} + \left(-\frac{u_- A_L \gamma}{(\delta+1)} (-y)^{-(\gamma+1)} + \gamma(\gamma+1) A_L y^{-(\gamma+2)}\right) t^{(\gamma\delta+2\delta+1)} + o\left(t^{(\gamma\delta+2\delta+1)}\right) \tag{2.67}$$

as $t \to \infty$ with $y = O(1)$. Expansion (2.67) becomes nonuniform when $y = O(t^{(\delta+\frac{1}{2})})$ as $t \to \infty$ (that is, when $x = O(t^{\frac{1}{2}})$ as $t \to \infty$]). To examine this region, which we label as region SS, we introduce the scaled coordinate $z$ by

$$z = x\,t^{-\frac{1}{2}} = y\,t^{-(\delta+\frac{1}{2})}, \tag{2.68}$$

as $t \to \infty$ with $z = O(1)$, and expand in the form

$$u(z,t) = U(z) + o(1) \tag{2.69}$$

as $t \to \infty$ with $z = O(1)$. On substituting (2.69) into equation (1.1) (when written in terms of $z$ and $t$) we find at leading order that

$$U_{zz} + \frac{z}{2}U_z = 0, \quad -\infty < z < \infty. \tag{2.70}$$

Equation (2.70) is to be solved subject to the matching conditions with region III$^+$ (as $z \to \infty$) and region III$^-$ (as $z \to -\infty$), namely,

$$U(z) \to \begin{cases} u_+, & \text{as} \quad z \to \infty, \\ u_-, & \text{as} \quad z \to -\infty. \end{cases} \tag{2.71}$$

The solution to (2.70) and (2.71) is then obtained as

$$U(z) = \frac{(u_+ + u_-)}{2} - \frac{(u_- - u_+)}{2}\mathrm{erf}\left(\frac{z}{2}\right), \quad -\infty < z < \infty, \tag{2.72}$$

where erf(.) is the standard error function (see for example [1]). It is straightforward to see that it is not possible to match expansion (2.69) (as $z \to \infty$) to expansion (2.66) and expansion (2.69) (as $z \to -\infty$) to expansion (2.67), beyond leading order. We will therefore need transition regions, which we label TR$^+$ and TR$^-$. To examine region TR$^+$ we introduce the scaled coordinate, $\eta$, via

$$z = c(t) + \frac{\eta}{(\ln t)^{\frac{1}{2}}} \tag{2.73}$$

where

$$c(t) = \sqrt{2\gamma}(\ln t)^{\frac{1}{2}} + \frac{(\gamma-1)}{\sqrt{2\gamma}}\frac{\ln(\ln t)}{(\ln t)^{\frac{1}{2}}} + o\left(\frac{\ln(\ln t)}{(\ln t)^{\frac{1}{2}}}\right) \tag{2.74}$$

as $t \to \infty$, and is determined so that matching between regions TR$^+$, III$^+$ and SS is possible, and look for an expansion of the form

$$u(\eta,t) = u_+ + F(\eta)(\ln t)^{-\frac{\gamma}{2}}t^{-\frac{\gamma}{2}} + o\left((\ln t)^{-\frac{\gamma}{2}}t^{-\frac{\gamma}{2}}\right) \tag{2.75}$$

as $t \to \infty$ with $\eta = O(1)$. The form of this expansion is dictated by the structure of expansions (2.66) and (2.69) in regions III$^+$ and SS, when written in region TR$^+$. It is instructive at this point to describe in more detail how (2.74) was obtained. Expansion (2.69) with (2.72) (for $z \gg 1$) when written in terms of $z$ is given by

$$u \sim u_+ + \frac{(u_- - u_+)}{\sqrt{\pi}z}\exp\left(-\frac{z^2}{4}\right) \tag{2.76}$$

as $t \to \infty$. Expansion (2.66) when written in terms of $z$ gives that

$$u \sim u_+ + \frac{A_R}{z^\gamma}t^{-\frac{\gamma}{2}} \tag{2.77}$$

as $t \to \infty$. Comparison of the corrections to $u_+$ in (2.76) and (2.77) indicates that they are of the same order when $z = c(t)$ as $t \to \infty$. On substituting expansion (2.75) into equation (1.1) (when written in terms of $\eta$ and $t$) we obtain at leading order that

$$F_{\eta\eta} + \frac{\sqrt{2\gamma}}{2}F_\eta = 0, \quad -\infty < \eta < \infty. \tag{2.78}$$

Equation (2.78) has to be solved subject to the matching conditions

$$F(\eta) \sim \begin{cases} \frac{A_R}{(2\gamma)^{\frac{\gamma}{2}}} & \text{as} \quad \eta \to \infty, \\ \frac{(u_- - u_+)}{\sqrt{2\gamma\pi}} e^{-\sqrt{\frac{\gamma}{2}}\eta} & \text{as} \quad \eta \to -\infty. \end{cases} \tag{2.79}$$

The solution to (2.78), (2.79) is readily obtained as

$$F(\eta) = \frac{A_R}{(2\gamma)^{\frac{\gamma}{2}}} + \frac{(u_- - u_+)}{\sqrt{2\gamma\pi}} e^{-\sqrt{\frac{\gamma}{2}}\eta}, \quad -\infty < \eta < \infty. \tag{2.80}$$

Therefore, the solution in region TR$^+$ is given by

$$u(\eta, t) = u_+ + \left( \frac{A_R}{(2\gamma)^{\frac{\gamma}{2}}} + \frac{(u_- - u_+)}{\sqrt{2\gamma\pi}} e^{-\sqrt{\frac{\gamma}{2}}\eta} \right) (\ln t)^{-\frac{\gamma}{2}} t^{-\frac{\gamma}{2}} + o\left( (\ln t)^{-\frac{\gamma}{2}} t^{-\frac{\gamma}{2}} \right) \tag{2.81}$$

as $t \to \infty$ with $\eta = O(1)$. The details of region TR$^-$ follow, after minor modification, those given for region TR$^+$ and are not repeated in full here. To examine region TR$^-$ we introduce the scaled coordinate, $\hat{\eta}$, via

$$z = -c(t) + \frac{\hat{\eta}}{(\ln t)^{\frac{1}{2}}} \tag{2.82}$$

as $t \to \infty$ with $\hat{\eta} = O(1)$, and look for an expansion of the form (2.75). The function $c(t)$ is given by (2.74). On substituting expansion (2.75) into equation (1.1) (when written in terms in $\hat{\eta}$ and $t$) we find (after satisfying the matching conditions with regions SS (as $\hat{\eta} \to \infty$) and III$^-$ (as $\hat{\eta} \to -\infty$)) that the solution in region TR$^-$ is given by

$$u(\hat{\xi}, t) = u_- + \left( \frac{A_L}{(2\gamma)^{\frac{\gamma}{2}}} - \frac{(u_- - u_+)}{\sqrt{2\gamma\pi}} e^{\sqrt{\frac{\gamma}{2}}\hat{\xi}} \right) (\ln t)^{-\frac{\gamma}{2}} t^{-\frac{\gamma}{2}} + o\left( (\ln t)^{-\frac{\gamma}{2}} t^{-\frac{\gamma}{2}} \right) \tag{2.83}$$

as $t \to \infty$ with $\hat{\xi} = O(1)$. It is worth noting here that higher order matching of expansion (2.81) in region TR$^+$ and expansion (2.83) in region TR$^-$ with expansion (2.69) in region SS requires the correction term in region SS to be $O\left(t^{-\frac{\gamma}{2}}(\ln t)^{-\frac{\gamma}{2}}\right)$.

The asymptotic structure of **IVP** as $t \to \infty$ when $-1 < \delta < -\frac{1}{2}$ and $\gamma \geq 1$ is now complete. A uniform approximation has been given through regions II$^\pm$, III$^\pm$, TR$^\pm$ and SS. A schematic representation of the location and thickness of the asymptotic regions as $t \to \infty$ is given for in Figure 5. The large-t attractor for the solution of **IVP** in this case is the error function which allows for the adjustment of the solution from $u_+$ to $u_-$. This attractor is in a stretching frame of reference of thickness $x = O(t^{\frac{1}{2}})$ as $t \to \infty$.

(b) $\mathbf{0 < \gamma < 1}$

In this case expansions (2.3) and (2.5) of regions II$^+$ ($x \to \infty$, $t = O(1)$) and II$^-$ ($x \to -\infty$, $t = O(1)$) respectively , continue to remain uniform provided $|x| \gg t^{\frac{\delta}{(\gamma-1)}}$ as $t \to \infty$. However, as already noted, a nonuniformity develops when $|x| = O(t^{\frac{\delta}{(\gamma-1)}})$. We begin by considering the asymptotic structure as $t \to \infty$ by moving in from region II$^+$ when $x \gg t^{\frac{\delta}{(\gamma-1)}}$ as $t \to \infty$. To proceed we introduce a new region, labelled as region III$^+$. To examine region III$^+$ we introduce the scaled coordinate

$$y = x\, t^{-\frac{\delta}{(\gamma-1)}}, \tag{2.84}$$

where $y = O(1)$ as $t \to \infty$, and look for an expansion of the form

$$u(y, t) = u_+ + F_0(y) t^{-\frac{\gamma\delta}{\gamma-1}} + F_1(y) t^{-\frac{2\delta}{\gamma-1}+1} + F_2(y) t^{-\frac{\delta(\gamma+2)}{\gamma-1}+1} + o\left( t^{-\frac{\delta(\gamma+2)}{\gamma-1}+1} \right) \tag{2.85}$$

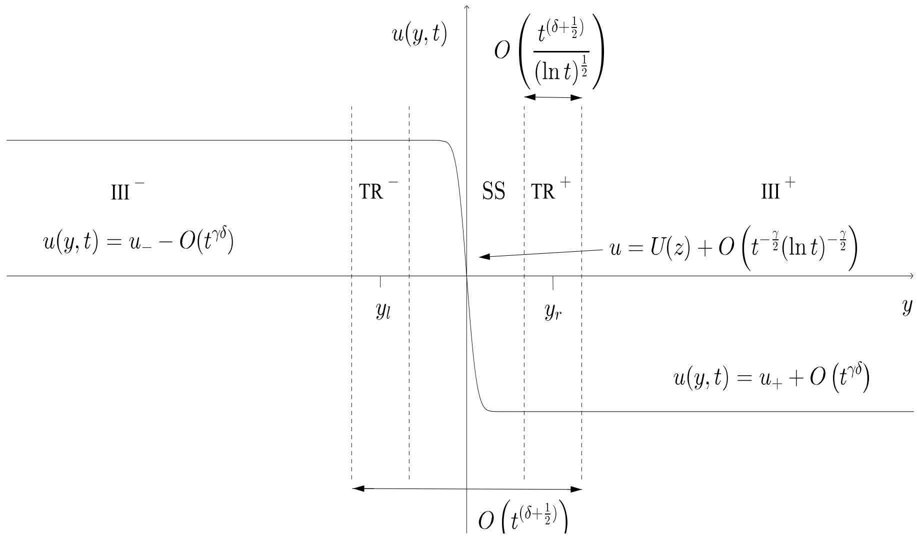


Figure 5: A schematic representation of the asymptotic structure of $u(y,t)$ in the $(y,u)$ plane as $t \to \infty$ for **IVP** when $-1 < \delta < -\frac{1}{2}$ and $\gamma \geq 1$. Here we illustrate the case when $u_+ < u_-$. We recall that $y_l = -y_r = -t^{\delta+\frac{1}{2}}c(t)$, where $c(t)$ is given by (2.74).

as $t \to \infty$ with $y = O(1)$, as indicated by the structure of expansion (2.3) in region II$^+$, when $x = O\left(t^{\frac{\delta}{(\gamma-1)}}\right)$. On substituting (2.85) into equation (1.1) (when written in terms of $y$ and $t$) and solving at each order in turn, we find (after matching with (2.3)) that

$$\begin{aligned} u(y,t) = u_+ + A_R y^{-\gamma} t^{-\frac{\gamma\delta}{(\gamma-1)}} + \frac{\gamma u_+ A_R}{(\delta+1)} y^{-(\gamma+1)} t^{-\frac{2\delta}{(\gamma-1)}+1} \\ + \left(\frac{\gamma A_R^2}{(\delta+1)} y^{-(2\gamma+1)} + \gamma(\gamma+1) A_R y^{-(\gamma+2)}\right) t^{-\frac{\delta(\gamma+2)}{(\gamma-1)}+1} \\ + o\left(t^{-\frac{\delta(\gamma+2)}{(\gamma-1)}+1}\right) \end{aligned} \tag{2.86}$$

as $t \to \infty$ with $y = O(1)$. Expansion (2.86) becomes nonuniform when $y = O(t^{-\frac{\delta\gamma}{(\gamma-1)}})$ as $t \to \infty$ (that is, when $x = O(t^{-\delta})$ as $t \to \infty$). To proceed we relabel region III$^+$ as region III$^+$(a) and introduce a new region, labelled as region III$^+$(b). To examine region III$^+$(b) we introduce the scaled coordinate

$$\hat{y} = y\, t^{\frac{\delta\gamma}{(\gamma-1)}}, \tag{2.87}$$

where $\hat{y} = O(1)$ as $t \to \infty$, and look for an expansion of the form

$$u(\hat{y},t) = u_+ + \widehat{F}_0(\hat{y}) t^{\gamma\delta} + \widehat{F}_1(\hat{y}) t^{(\gamma\delta+2\delta+1)} + o\left(t^{(\gamma\delta+2\delta+1)}\right) \tag{2.88}$$

as $t \to \infty$ with $\hat{y} = O(1)$. On substituting (2.88) into equation (1.1) (when written in terms of $y$ and $t$) and solving at each order in turn, we find that

$$\begin{aligned} u(\hat{y},t) = u_+ + A_R \hat{y}^{-\gamma} t^{\gamma\delta} + \left(\frac{u_+ A_R \gamma}{(\delta+1)} \hat{y}^{-(\gamma+1)} + \gamma(\gamma+1) A_R \hat{y}^{-(\gamma+2)}\right) t^{(\gamma\delta+2\delta+1)} \\ + o\left(t^{(\gamma\delta+2\delta+1)}\right) \end{aligned} \tag{2.89}$$

as $t \to \infty$ with $\hat{y} = O(1)$. Expansion (2.89) becomes nonuniform when $\hat{y} = O\left(t^{(\delta+\frac{1}{2})}\right)$ as $t \to \infty$ (that is, when $x = O(t^{\frac{1}{2}})$ as $t \to \infty$). The remaining asymptotic structure in this case now follows that given in Section 2.3.3(a) for $\gamma \geq 1$ and is not repeated here. We note that region III$^-$ of Section 2.3.3(a) will need to be replaced by region III$^-$(a) and region III$^-$(b). The details of these regions follow, after minor modification, those given for regions III$^+$(a) and III$^+$(b) above. The asymptotic structure of **IVP** as $t \to \infty$ when $-1 < \delta < -\frac{1}{2}$ and $0 < \gamma < 1$ is now complete. A uniform approximation has been given through regions II$^\pm$, III(a)$^\pm$, III$^\pm$(b), TR$^\pm$ and SS. The large-t attractor for the solution of **IVP** in this case is the error function which allows for the adjustment of the solution from $u_+$ to $u_-$. This attractor is in a stretching frame of reference of thickness $x = O(t^{\frac{1}{2}})$ as $t \to \infty$. We can summarize the results of this section in,

**Proposition 3.** *Let $u : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ be the solution to* **IVP** *when $\gamma > 0$, $-1 < \delta < -\frac{1}{2}$ and any $u_+$ and $u_-$. In terms of the coordinate $\bar{y} = \frac{x}{t^{\frac{1}{2}}}$, on writing,*

$$u(\bar{y}, t) = \left( \frac{(u_+ + u_-)}{2} - \frac{(u_- - u_+)}{2} \mathit{erf}(\bar{y}) \right) + R(\bar{y}, t)$$

*for $(\bar{y}, t) \in \mathbb{R} \times [0, \infty)$, then $R(\bar{y}, t) \to 0$ as $t \to \infty$ uniformly for $\bar{y} \in \mathbb{R}$. In particular, for $\gamma \geq 1$,*

$$R(\bar{y}, t) = \begin{cases} O\left( \frac{t^{\gamma\delta}}{(1+|y|^\gamma)} \right) & \text{in regions } III^\pm, \\ O\left( t^{-\frac{\gamma}{2}} (\ln t)^{-\frac{\gamma}{2}} \right) & \text{in regions } TR^\pm, \\ O\left( t^{-\frac{\gamma}{2}} (\ln t)^{-\frac{\gamma}{2}} \right) & \text{in region } SS, \end{cases}$$

*as $t \to \infty$, whilst for $0 < \gamma < 1$,*

$$R(\bar{y}, t) = \begin{cases} O\left( \frac{t^{\frac{\gamma\delta}{(1-\gamma)}}}{(1+|y|^\gamma)} \right) & \text{in regions } III(a)^\pm, \\ O\left( t^{\gamma\delta} \right) & \text{in regions } III(b)^\pm, \\ O\left( t^{-\frac{\gamma}{2}} (\ln t)^{-\frac{\gamma}{2}} \right) & \text{in regions } TR^\pm, \\ O\left( t^{-\frac{\gamma}{2}} (\ln t)^{-\frac{\gamma}{2}} \right) & \text{in region } SS, \end{cases}$$

*as $t \to \infty$.*

### 2.3.4 $\delta = -\frac{1}{2}$

We recall from Section 2.2 that in this case we need to consider the cases $\gamma \geq 1$ and $0 < \gamma < 1$ separately. We begin by considering the case $0 < \gamma < 1$.

(a) $\mathbf{0 < \gamma < 1}$

In this case expansions (2.3) and (2.5) of region II$^+$ ($x \to \infty$, $t = O(1)$) and region II$^-$ ($x \to -\infty$, $t = O(1)$) respectively , continue to remain uniform provided $|x| \gg t^{\frac{1}{2(1-\gamma)}}$ as $t \to \infty$. However, as already noted a nonuniformity develops when $|x| = O(t^{\frac{1}{2(1-\gamma)}})$. We begin by considering the asymptotic structure as $t \to \infty$ moving in from region II$^+$, when $x \gg t^{\frac{1}{2(\gamma-1)}}$ as $t \to \infty$. To proceed we introduce region III$^+$. To examine region III$^+$ we introduce the scaled coordinate

$$y = x\,t^{-\frac{1}{2(1-\gamma)}}, \tag{2.90}$$

where $y = O(1)$ as $t \to \infty$, and look for an expansion of the form

$$u(y, t) = u_+ + F_0(y) t^{-\frac{\gamma}{2(1-\gamma)}} + F_1(y) t^{-\frac{\gamma}{(1-\gamma)}} + F_2(y) t^{-\frac{3\gamma}{2(1-\gamma)}} + o\left( t^{-\frac{3\gamma}{2(1-\gamma)}} \right) \tag{2.91}$$

as $t \to \infty$ with $y = O(1)$. On substituting (2.91) into equation (1.1) and solving at each order in turn, we find (after matching with (2.3)) that

$$\begin{aligned} u(y,t) = & u_+ + A_R y^{-\gamma} t^{-\frac{\gamma}{2(1-\gamma)}} + 2u_+ \gamma A_R y^{-\gamma-1} t^{-\frac{\gamma}{(1-\gamma)}} \\ & + \left( 2\gamma A_R^2 y^{-(2\gamma+1)} + \left( \gamma(\gamma+1)A_R + 2\gamma(\gamma+1)u_+^2 A_R \right) y^{-(\gamma+1)} \right) t^{-\frac{3\gamma}{2(1-\gamma)}} \\ & + o\left( t^{-\frac{3\gamma}{2(1-\gamma)}} \right) \end{aligned} \tag{2.92}$$

as $t \to \infty$ with $y = O(1)\,(>0)$. We observe that expansion (2.92) becomes nonuniform when $y = O\left(t^{-\frac{\gamma}{2(1-\gamma)}}\right)$ as $t \to \infty$ (that is, when $x = O\left(t^{\frac{1}{2}}\right)$ as $t \to \infty$). We next consider the asymptotic structure as $t \to \infty$ moving in from region II$^-$ when $(-x) \gg t^{\frac{1}{2(\gamma-1)}}$. To proceed we introduce region III$^-$. The details of this region follow, after minor modification, those given for region III$^+$ and are not repeated here. In region III$^-$ we have that

$$u(y,t) = u_- + A_L(-y)^{-\gamma} t^{-\frac{\gamma}{2(1-\gamma)}} + o\left(t^{-\frac{\gamma}{2(1-\gamma)}}\right) \tag{2.93}$$

as $t \to \infty$ with $y = O(1)(<0)$. Again we observe that expansion (2.93) becomes nonuniform when $y = O\left(t^{-\frac{\gamma}{2(1-\gamma)}}\right)$ as $t \to \infty$ (that is, when $x = O(t^{\frac{1}{2}})$ as $t \to \infty$). To examine this region, which we label region SS, we introduce the scaled variable

$$z = xt^{-\frac{1}{2}} = yt^{\frac{\gamma}{2(1-\gamma)}}$$

and look for an expansion of the form

$$u(z,t) = U(z) + o(1) \tag{2.94}$$

as $t \to \infty$ with $z = O(1)$. On substituting expansion (2.94) into equation (1.1) (when written in terms of $z$ and $t$) we obtain at leading order that

$$U_{zz} + \left(\frac{z}{2} - U\right) U_z = 0, \quad -\infty < z < \infty. \tag{2.95}$$

Equation (2.95) is to be solved subject to the matching conditions with region III$^+$ ($z \to \infty$) and region III$^-$ ($z \to -\infty$), namely,

$$U(z) \to \begin{cases} u_+ & \text{as} \quad z \to \infty, \\ u_- & \text{as} \quad z \to -\infty. \end{cases} \tag{2.96}$$

The boundary value problem (2.95) and (2.96) has been examined by Rudenko and Soluyan [6] and Scott [7]. It was established that for each $u_+$ and $u_-$ ($u_+ \neq u_-$) then (2.95) and (2.96) has a unique solution $U = u_R(z)$, which is strictly monotone in $z$, and satisfies,

$$u_R(z) \sim \begin{cases} u_+ + \mathcal{C}_+(u_+,u_-) z^{-1} e^{-\frac{1}{4}(z-2u_+)^2} & \text{as} \quad z \to \infty, \\ u_- + \mathcal{C}_-(u_+,u_-) z^{-1} e^{-\frac{1}{4}(z-2u_-)^2} & \text{as} \quad z \to -\infty, \end{cases} \tag{2.97}$$

with $\mathcal{C}_+$ and $\mathcal{C}_-$ globally determined nonzero constants, depending upon $u_+$ and $u_-$, with

$$\mathcal{C}_+(u_+,u_-) \begin{cases} > 0, & u_+ < u_-, \\ < 0, & u_+ > u_-, \end{cases} \quad \text{and} \quad \mathcal{C}_-(u_+,u_-) \begin{cases} < 0, & u_+ < u_-, \\ > 0, & u_+ > u_-. \end{cases} \tag{2.98}$$

However, we observe from (2.94) with (2.97) that expansion (2.94) fails to match at higher order to expansion (2.92) as $z \to \infty$ and (2.93 ) as $z \to -\infty$. We therefore require transition regions $\mathrm{TR}^{\pm}$. To examine region $\mathrm{TR}^{+}$ we introduce the scaled coordinate $\xi$, via

$$z = c(t) + \frac{\xi}{(\ln t)^{\frac{1}{2}}} \tag{2.99}$$

where

$$c(t) = (2\gamma)^{\frac{1}{2}}(\ln t)^{\frac{1}{2}} + 2u_+ + \frac{(\gamma-1)}{(2\gamma)^{\frac{1}{2}}}\frac{\ln(\ln t)}{(\ln t)^{\frac{1}{2}}} + o\left(\frac{\ln(\ln t)}{(\ln t)^{\frac{1}{2}}}\right) \tag{2.100}$$

as $t \to \infty$, and expand as

$$u(\xi,t) = u_+ + K(\xi)\,(\ln t)^{-\frac{\gamma}{2}}\,t^{-\frac{\gamma}{2}} + o\left((\ln t)^{-\frac{\gamma}{2}}\,t^{-\frac{\gamma}{2}}\right) \tag{2.101}$$

as $t \to \infty$ with $\xi = O(1)$. On substitution of expansion (2.101) into equation (1.1) (when written in terms of $\xi$ and $t$) we obtain at leading order

$$K_{\xi\xi} + \left(\frac{\gamma}{2}\right)^{\frac{1}{2}} K_{\xi} = 0, \quad -\infty < \xi < \infty. \tag{2.102}$$

Equation (2.102) is to be solved subject to the matching conditions with region $\mathrm{III}^{+}$ ($\xi \to \infty$) and region SS ($\xi \to -\infty$), namely,

$$K(\xi) \sim \begin{cases} \mathcal{C}_+(u_+,u_-)(2\gamma)^{-\frac{1}{2}}e^{-\left(\frac{\gamma}{2}\right)^{\frac{1}{2}}\xi} & \text{as} \quad \xi \to -\infty, \\ A_R(2\gamma)^{-\frac{\gamma}{2}} & \text{as} \quad \xi \to \infty. \end{cases} \tag{2.103}$$

The expansion in region $\mathrm{TR}^{+}$ is then given by

$$u(\xi,t) = u_+ + \left(\mathcal{C}_+(u_+,u_-)(2\gamma)^{-\frac{1}{2}}e^{-\left(\frac{\gamma}{2}\right)^{\frac{1}{2}}\xi} + A_R(2\gamma)^{-\frac{\gamma}{2}}\right)(\ln t)^{-\frac{\gamma}{2}}\,t^{-\frac{\gamma}{2}} + o\left((\ln t)^{-\frac{\gamma}{2}}\,t^{-\frac{\gamma}{2}}\right) \tag{2.104}$$

as $t \to \infty$ with $\xi = O(1)$. To examine region $\mathrm{TR}^{-}$ we introduce the scaled coordinate, $\hat{\xi}$, via

$$z = -c(t) + \frac{\hat{\xi}}{(\ln t)^{\frac{1}{2}}} \tag{2.105}$$

as $t \to \infty$ with $\hat{\xi} = O(1)$, and look for an expansion of the form (2.101). The function $c(t)$ is given by (2.100) with $u_+$ replaced by $u_-$. On substituting expansion (2.101) into equation (1.1) (when written in terms in $\hat{\xi}$ and $t$) we find (after satisfying the matching conditions with regions SS (as $\hat{\xi} \to \infty$) and $\mathrm{III}^{-}$ (as $\hat{\xi} \to -\infty$)) that the solution in region $\mathrm{TR}^{-}$ is given by

$$u(\hat{\xi},t) = u_- + \left(\frac{A_L}{(2\gamma)^{\frac{\gamma}{2}}} + \mathcal{C}_-(u_+,u_-)(2\gamma)^{-\frac{1}{2}}e^{\left(\frac{\gamma}{2}\right)^{\frac{1}{2}}\hat{\xi}}\right)(\ln t)^{-\frac{\gamma}{2}}t^{-\frac{\gamma}{2}} + o\left((\ln t)^{-\frac{\gamma}{2}}t^{-\frac{\gamma}{2}}\right) \tag{2.106}$$

as $t \to \infty$ with $\hat{\xi} = O(1)$. It is again worth noting that higher order matching of expansions (2.104) and (2.106) in regions $\mathrm{TR}^{+}$ and $\mathrm{TR}^{-}$, with expansion (2.94) in region SS requires the correction term in region SS to be $O\left(t^{-\frac{\gamma}{2}}(\ln t)^{-\frac{\gamma}{2}}\right)$.

The asymptotic structure of **IVP** as $t \to \infty$ when $\delta = -\frac{1}{2}$ and $0 < \gamma < 1$ is now complete. A uniform approximation has been given through regions $\mathrm{II}^{\pm}$, $\mathrm{III}^{\pm}$, $\mathrm{TR}^{\pm}$ and SS. A schematic representation of the location and thickness of the asymptotic regions as $t \to \infty$ is given for in Figure 6. The large-t attractor for the solution of **IVP** in this case is the similarity solution found by Rudenko and Soluyan in [6], which allows for the adjustment of the solution from $u_+$ to $u_-$. This attractor is in a stretching frame of reference of thickness $x = O(t^{\frac{1}{2}})$ as $t \to \infty$.

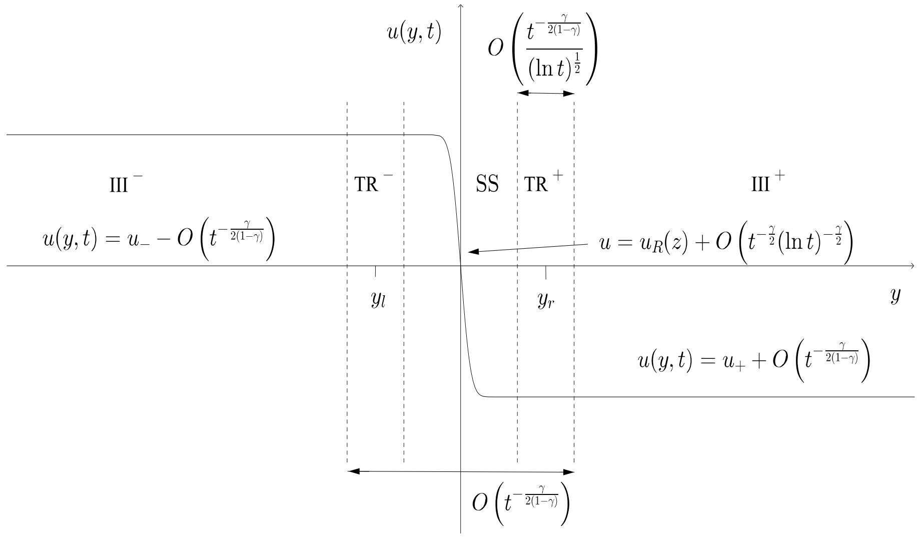


Figure 6: A schematic representation of the asymptotic structure of $u(y,t)$ in the $(y,u)$ plane as $t \to \infty$ for **IVP** when $\delta = -\frac{1}{2}$ and $0 < \gamma < 1$. Here we illustrate the case when $u_+ < u_-$. We recall that $y_l = -y_r = -t^{-\frac{\gamma}{2(1-\gamma)}} c(t)$, where $c(t)$ is given by (2.100).

(b) $\boldsymbol{\gamma \geq 1}$

In this case expansions (2.3) and (2.5) of regions II$^+$ ($x \to \infty$, $t = O(1)$) and II$^-$ ($x \to -\infty$, $t = O(1)$) respectively , continue to remain uniform provided $|x| \gg t^{\frac{1}{2}}$. However, as already noted a nonuniformity develops when $|x| = O(t^{\frac{1}{2}})$ as $t \to \infty$. We have established in Section 2.3.4(a) that the solution in region SS, where $|x| = O(t^{\frac{1}{2}})$ as $t \to \infty$, is given by expansion (2.94) with the similarity solution $U = u_R(z)$. Therefore, all that remains in this case is to introduce transition regions TR$^\pm$ to allow the solution in region SS to match to the far field where $|x| \gg t^{\frac{1}{2}}$ as $t \to \infty$. The details of regions TR$^\pm$ follow those given in Section 2.3.4(a) and are not repeated here. The asymptotic structure of **IVP** as $t \to \infty$ when $\delta = -\frac{1}{2}$ and $\gamma \geq 1$ is now complete. A uniform approximation has been given through regions II$^\pm$, TR$^\pm$ and SS. The large-$t$ attractor for the solution of **IVP** in this case is the similarity solution found by Rudenko and Soluyan in [6], which allows for the adjustment of the solution from $u_+$ to $u_-$. This attractor is in a stretching frame of reference of thickness $|x| = O(t^{\frac{1}{2}})$ as $t \to \infty$.

We can summarize the results of this section in,

**Proposition 4.** *Let $u : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ be the solution to* **IVP** *when $\gamma > 0$, $\delta = -\frac{1}{2}$ and any $u_+$ and $u_-$. In terms of the coordinate $\bar{y} = \frac{x}{t^{\frac{1}{2}}}$, on writing,*

$$u(\bar{y}, t) = u_R(\bar{y}) + R(\bar{y}, t)$$

*for $(\bar{y}, t) \in \mathbb{R} \times [0, \infty)$, then $R(\bar{y}, t) \to 0$ as $t \to \infty$ uniformly for $\bar{y} \in \mathbb{R}$. In particular, for $0 < \gamma < 1$, as $t \to \infty$,*

$$R(\bar{y}, t) = \begin{cases} O\left(\frac{t^{-\frac{\gamma}{2(1-\gamma)}}}{(1+|y|^\gamma)}\right) & \text{in regions } III^\pm, \\ O\left(t^{-\frac{\gamma}{2}}(\ln t)^{-\frac{\gamma}{2}}\right) & \text{in regions } TR^\pm, \\ O\left(t^{-\frac{\gamma}{2}}(\ln t)^{-\frac{\gamma}{2}}\right) & \text{in region } SS, \end{cases}$$

*where* $y = \bar{y}\, t^{-\frac{\gamma}{2(1-\gamma)}}$, *whilst for* $\gamma \geq 1$, *as* $t \to \infty$,

$$R(\bar{y}, t) = \begin{cases} O\left(\frac{t^{-\frac{\gamma}{2}}}{(1+|y|^{\gamma})}\right) & \textit{in regions } III^{\pm}, \\ O\left(t^{-\frac{\gamma}{2}}(\ln t)^{-\frac{\gamma}{2}}\right) & \textit{in regions } TR^{\pm}, \\ O\left(t^{-\frac{\gamma}{2}}(\ln t)^{-\frac{\gamma}{2}}\right) & \textit{in region } SS, \end{cases}$$

# 3 Summary

In this paper we have obtained, via the method of matched asymptotic coordinate expansions, the uniform asymptotic structure of the large-$t$ solution to the initial-value problem **IVP** over all parameter values. The form of the large-$t$ solution of initial-value problem **IVP** depends on the problem parameters $\delta$, $\gamma$, $u_+$ and $u_-$ as follows:

(i) When $\delta > -\frac{1}{2}$, $u_+ > u_-$ and $\gamma > 0$, the solution $u(x,t)$ to **IVP** exhibits the formation of an expansion wave profile, with

$$u(yt^{(\delta+1)}, t) \to \begin{cases} u_+, & y > \frac{u_+}{(\delta+1)}, \\ (\delta+1)y, & \frac{u_-}{(\delta+1)} \leq y \leq \frac{u_+}{(\delta+1)}, \\ u_-, & y < \frac{u_-}{(\delta+1)}, \end{cases}$$

as $t \to \infty$, uniformly for $y \in \mathbb{R}$. The detailed rate of convergence is given in Proposition 1.

(ii) When $\delta > -\frac{1}{2}$, $u_+ < u_-$ and $\gamma > 0$, the solution $u(x,t)$ to **IVP** exhibits the formation of a localized Taylor shock profile, with

$$u\left(\frac{(u_+ + u_-)}{2(\delta+1)} t^{(\delta+1)} + zt^{-\delta}, t\right) \to \left[\frac{(u_+ + u_-)}{2} - \frac{(u_- - u_+)}{2} \tanh\left(\frac{(u_- - u_+)}{4} z + \phi_c\right)\right]$$

as $t \to \infty$ with $z = O(1)$, and $\phi_c$ being a globally determined constant. It follows that the Taylor shock wave-front is at $x = s(t)$, where

$$s(t) = \frac{(u_+ + u_-)}{2(\delta+1)} t^{(\delta+1)} + c\, t^{-\delta} + o\left(t^{-\delta}\right) \tag{3.1}$$

as $t \to \infty$, with $c$ being a globally determined constant, which is a consequence of the evolution over all $t \geq 0$, and is undetermined by our asymptotic analysis as $t \to \infty$. The Taylor shock propagation speed is then

$$\dot{s} = \frac{(u_+ + u_-)}{2} t^{\delta} - \delta\, c\, t^{-(\delta+1)} + o\left(t^{-(\delta+1)}\right)$$

as $t \to \infty$. We see that,

(a) When $u_+ = -u_-$, the the Taylor shock front is decelerating as $t \to \infty$ with $\dot{s}(t) = O\left(t^{-(\delta+1)}\right)$ as $t \to \infty$, whilst

$$s(t) \to \begin{cases} 0, & \delta > 0, \\ c, & \delta = 0 \end{cases}$$

as $t \to \infty$, and

$$s(t) \sim ct^{|\delta|}$$

as $t \to \infty$ when $-\frac{1}{2} < \delta < 0$. In each case the Taylor shock profile is contained within a region of thickness $O(t^{-\delta})$, which is a thinning region when $\delta > 0$ and a thickening region when $-\frac{1}{2} < \delta < 0$.

(b) When $u_+ < -u_-$, the the Taylor shock front has negative acceleration as $t \to \infty$ when $\delta > 0$ ($\dot{s}(t) \to -\infty$ as $t \to \infty$), but negative deceleration when $-\frac{1}{2} < \delta < 0$ ($\dot{s} \to 0^-$ as $t \to \infty$). However, $s(t) \to -\infty$ as $t \to \infty$ for all $\delta > -\frac{1}{2}$. The region containing the Taylor shock front is of thickness $O(t^{-\delta})$ in each case.

(c) When $u_+ > -u_-$, the the Taylor shock front has positive acceleration as $t \to \infty$ when $\delta > 0$ ($\dot{s}(t) \to \infty$ as $t \to \infty$), but positive deceleration when $-\frac{1}{2} < \delta < 0$ ($\dot{s} \to 0^+$ as $t \to \infty$). However, $s(t) \to \infty$ as $t \to \infty$ for all $\delta > -\frac{1}{2}$. Again the region containing the Taylor shock front is of thickness $O(t^{-\delta})$ in each case.

(iii) When $-1 < \delta < -\frac{1}{2}$, any $u_+$ and $u_-$ and $\gamma > 0$, the solution $u(x,t)$ to **IVP** exhibits the formation of an error function profile, where

$$u\left(zt^{\frac{1}{2}}, t\right) \to \left[\frac{(u_+ + u_-)}{2} - \frac{(u_- - u_+)}{2} \operatorname{erf}\left(\frac{z}{2}\right)\right]$$

as $t \to \infty$ with $z = O(1)$. We observe that error function profile is in a stretching frame of reference of thickness $O(t^{\frac{1}{2}})$ as $t \to \infty$.

(iv) When $\delta = -\frac{1}{2}$, any $u_+$ and $u_-$ and $\gamma > 0$, the solution $u(x,t)$ to **IVP** exhibits the formation of the similarity solution found by Rudenko and Soluyan [6], where $u(zt^{\frac{1}{2}}, t) \to u_R(z)$ as $t \to \infty$ with $z = O(1)$. We observe that this profile is in a stretching frame of reference of thickness $O(t^{\frac{1}{2}})$ as $t \to \infty$.

Finally, it is interesting to consider a specific application. We consider the dynamics of a linearly and weakly compressible viscous fluid, with time dependent viscosity, in one spatial dimension. With $\bar{u}$ being the fluid velocity field, the unidirectional evolution of initial finite amplitude disturbances is governed by the following Burgers' equation,

$$\bar{u}_t + \left(c_0 + \frac{\mu'(t)}{2\rho_0 c_0} + \frac{1}{2}\bar{u}\right)\bar{u}_x = \frac{\mu(t)}{2\rho_0}\bar{u}_{xx} \tag{3.2}$$

as $t > 0$ being time and $x \in \mathbb{R}$ being the spatial coordinate, whilst $c_0$ is the linearly compressive sound speed, $\rho_0$ is the ambient fluid density and $\mu(t)$ is the fluid viscosity. In relation to the current paper, we give attention to the case where the fluid viscosity is time dependent, through, for example, increase or decrease in ambient temperature. In particular, we put,

$$\mu = \mu(t) = \mu_0 t^{\lambda} \tag{3.3}$$

with $\mu_0 > 0$ and $\lambda > -1$. Thus the fluid viscosity is increasing with $t > 0$ when $\lambda > 1$, but decreasing with $t > 0$ when $-1 < \lambda < 0$. Now introduce the simple transformation,

$$\begin{aligned} z &= x - \left(c_0 t + \frac{\mu_0 t^{\lambda}}{2\rho_0 c_0}\right), \\ \tau &= \frac{\mu_0}{2(1+\lambda)\rho_0} t^{(1+\lambda)}, \\ u &= (2(1+\lambda))^{-\frac{\lambda}{(1+\lambda)}} \left(\frac{\rho_0}{\mu_0}\right)^{\frac{1}{(1+\lambda)}} \bar{u}. \end{aligned} \tag{3.4}$$

On substitution from (3.4), the partial differential equation (3.2) becomes

$$u_\tau + \tau^{-\frac{\lambda}{(1+\lambda)}} u u_z = u_{zz}, \quad -\infty < z < \infty, \quad \tau > 0, \tag{3.5}$$

which is now in the form of the generalized Burgers' equation considered in this paper, with, for each $\lambda \in (-1, \infty)$,

$$\delta = -\frac{\lambda}{(1+\lambda)} \in (-1, \infty). \tag{3.6}$$

We observe from (3.6) that, $\delta$ is a monotone decreasing function of $\lambda$, with $\delta \to \infty$ as $\lambda \to -1^+$ and $\delta \to -1$ as $\lambda \to \infty$. In fact,

$$\begin{aligned} &\delta \in \left(-1, -\frac{1}{2}\right) \quad \text{when} \quad \lambda \in (1, \infty), \\ &\delta = -\frac{1}{2} \quad \text{when} \quad \lambda = 1, \\ &\delta \in \left(-\frac{1}{2}, \infty\right) \quad \text{when} \quad \lambda \in (-1, 1). \end{aligned} \tag{3.7}$$

For transition-type initial conditions the theory developed in this paper, and in particular cases (i)-(iv) above, determine the detailed evolution of (3.5). Principally we observe that when $\lambda \in (1, \infty)$ an error function profile will develop as $\tau \to \infty$, for any $u_+ \lesseqgtr u_-$. However, for $\lambda = 1$, the Rudenko-Soluyan similarity profile will develop as $\tau \to \infty$, for any $u_+ \lesseqgtr u_-$. However, for $\lambda \in (-1, 1)$, a Taylor shock profile will develop as $\tau \to \infty$, when $u_+ < u_-$, but an expansion wave will develop as $\tau \to \infty$, when $u_+ > u_-$.

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# References

[1] M. Abramowitz and I. Stegun. *Handbook of Mathematical Functions.* Dover (1965).

[2] E. Hanaç. *The Large-Time Solution of Nonlinear Evolution Equations* Ph.D. thesis, University of Birmingham (2015)

[3] J.A. Leach. The large-time solution of Burgers' equation with variable coefficients. I. Exponential coefficients. (To appear: *Stud. Appl. Math.*)

[4] J.A. Leach and D.J. Needham. Matched Asymptotic Expansions in Reaction-Diffusion Theory. Springer Monographs in Mathematics (2003).

[5] J.A. Leach and D.J. Needham. The large-time development of the solution to an initial-value problem for the Korteweg-de Vries equation: I. Initial data has a discontinuous expansive step. *Nonlinearity* 21, 2391-2408 (2008).

[6] O.V. Rudenko and S.I. Soluyan. Theoretical foundations of nonlinear acoustics (English translation by R. T. Beyer). Consultants Bureau, Plenum. (1977).

[7] J.F. Scott. The long time asymptotics of the solutions to the generalized Burgers equation. *Proc. R. Soc. Lond.* A373, 443-456 (1981).

[8] G.B. Whitham. Linear and Nonlinear Waves. Wiley, New York (1974).

[9] M. Van Dyke. Perturbation Methods in Fluid Dynamics. Parabolic Press, Stanford, CA, (1975).